MTH 2310, LINEAR ALGEBRA
DR. ADAM GRAHAM-SQUIRE, FALL 2017

TEST 3 REVIEW KEY

(1) True/False: If True, justify your answer with a brief explanation. If False, give a counterexample or a brief explanation.
   (a) A vector space is also a subspace. TRUE, it is a subspace of itself. That is, it satisfies the three subspace properties because it also satisfies them as part of the definition of vector space.
   (b) The column space Col A is not affected by elementary row operations on A. FALSE, after row reducing we need to take vectors from the original columns of a matrix because row operations can change the column space.
   (c) A linearly independent set in a subspace H is a basis for H. FALSE, it also needs to span.
   (d) Let $\mathcal{B}$ be a basis for V and $P_{\mathcal{B}}$ the change-of-coordinates matrix. Then $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}\mathbf{x}$ for all $\mathbf{x}$ in V. FALSE, should be $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.
   (e) The number of free variables in the equation $A\mathbf{x} = \mathbf{0}$ equals the dimension of Nul A. TRUE, when you do parametric vector form, the number of basis vectors you get for Nul A will be the same as the number of free variables.
   (f) If A and B are row equivalent, then their row spaces are the same. TRUE, row operations do not affect the linear dependence relations between rows.
   (g) The eigenvalues of a matrix are on its main diagonal. FALSE, this is only true when the matrix is a triangular matrix.
   (h) A row replacement operation on a matrix A does not change the eigenvalues. **False**, row operations can change the eigenvalues of a matrix.
   (i) If A is invertible, then A is diagonalizable. **False**, The 2 by 2 matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible but not diagonalizable.

(2) Let W be the set of all vectors of the form $\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}$. Is W a subspace of $\mathbb{R}^4$? Explain your reasoning.

   **Ans: Yes. Write the vector in parametric vector form, and you will get that all vectors of the form $\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}$ are in the span of certain vectors. The span of some vectors is always a subspace. (You could also check the properties of subspace: zero, addition, multiplication to see that it will be a subspace, or write it as a column space which is also a subspace. There are many ways to argue this one).

(3) Consider the following two systems of equations:

$$\begin{array}{rrrl} 5x_1+ & x_2- & 3x_3 = & 0 \\ -9x_1+ & 2x_2+ & 5x_3 = & 1 \\ 4x_1+ & x_2- & 6x_3 = & 9 \end{array} \quad \text{and} \quad \begin{array}{rrrl} 5x_1+ & x_2- & 3x_3 = & 0 \\ -9x_1+ & 2x_2+ & 5x_3 = & 5 \\ 4x_1+ & x_2- & 6x_3 = & 45 \end{array}$$

It can be shown that the first system has a solution. Use this fact to explain why the second system also has a solution without making any row operations.

**Ans: The first matrix says that $\begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$ is in the column space of $\begin{bmatrix} 5 & 1 & -3 \\ -9 & 2 & 5 \\ 4 & 1 & -6 \end{bmatrix}$. Since the column space is a subspace, every scalar multiple of $\begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$ is also in the column space, hence $\begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix}$ is in the column space, so the second system must have a solution as well (in fact it will be five times the solution to the first system, but that is not necessary).

(4) Consider the polynomials $\mathbf{p}_1(t) = 1 + t$, $\mathbf{p}_2(t) = 1 - t$, and $\mathbf{p}_3(t) = 2$. By inspection, write a linear dependence relation among $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$.

**Ans: $p_1 + p_2 - p_3 = 0$

Now find a basis for Span$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

**Ans: Write the polynomials as vectors and form them into the matrix $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$, when you row reduce you should see that the first two vectors will form a basis (although any two of the vectors will form a basis, because by inspection any two are linearly independent because none of them are scalar multiples).

(5) Are the polynomials $(1 - t)^3$, $(2 - 3t)2$, and $3t^2 - 4t^3$ linearly independent in $\mathbb{P}_3$? Do they form a basis?

**Ans: Expand each expression and write the results as vectors with four entries. Reduce the matrix to see if the columns are linearly independent (They will not form a basis in any case because you would need four vectors to span, and you only have three). They should reduce to $\begin{bmatrix} 1 & 4 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the vectors are in fact <u>not</u> linearly independent.

(6) Consider the following subspace of $\mathbb{R}^4$: $\left\{ \begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix} \right\}$. Find a basis for the subspace and state its dimension.

**Ans: Write the vector in parametric vector form to get vectors that span. You will still need to row reduce it to make sure that the vectors are linearly independent, and when you do that you should see that the third vector is linearly dependent on the first two. Thus the first two vectors are a basis and the dimension is 2.

(7) Let H be an n-dimensional subspace of an n-dimensional vector space V. Explain why $H = V$.

**Ans: A basis for H will have n vectors in it, so that is n linearly independent vectors that span H. By a theorem we mentioned in class, any n linearly independent vectors will also span an n dimensional space (because if you put them into a matrix you will have a pivot in every column, thus a pivot in every row as well), and since V is n dimensional, the basis vectors of H must also span V, thus they will be a basis for V as well. So H and V must be the same space.

(8) Is it possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Explain your answer in terms of rank as well as the dimension of things such as the column, null, and/or row spaces.

**Ans: No. One fixed solution means the dimension of the null space is 1, so the rank would have to be 11(because 12-1=11). This is not possible because our matrix would only have 10 rows (because there are ten equations).

(9) Find a basis for the eigenspace of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = -2$.

**Ans: Row reduce $\begin{bmatrix} 1+2 & 0 & -1 \\ 1 & -3+2 & 0 \\ 4 & -13 & 1+2 \end{bmatrix}$ and write your answer in parametric vector form. You should get one vector that is a scalar multiple of $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, so a basis for the eigenspace is just $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ (or whatever scalar multiple of that you got).

(10) Explain why A and A^T have the same characteristic polynomial (assume A is a square matrix).

Ans: Because λI is a diagonal matrix, it is equal to its transpose. Thus $A^T - \lambda I = A^T - (\lambda I)^T = (A - \lambda I)^T$ because the sum of transposes is the transpose of the sum (property of transpose). The characteristic polynomial of A^T is $\det(A^T - \lambda I)$, so we have
$\det(A^T - \lambda I) = \det(A - \lambda I)^T = \det(A - \lambda I)$ = char. polyn. of A
because the determinant of a matrix and its transpose are the same.

(11) Diagonalize $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

Ans: $D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$ and $P = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}$. Note: There are many answers for P, and you can order D differently as well and still have the correct answer.